Black-hole perturbation theory: The asymptotic spectrum of the prolate spin-weighted spheroidal harmonics

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Prolate spin-weighted spheroidal harmonics play a key role in black-hole perturbation theory. In particular, the highly damped quasinormal resonances of rotating Kerr black holes are closely related to the asymptotic eigenvalues of these important functions. We here present a novel and compact derivation of the asymptotic eigenvalues of the prolate spin-weighted spheroidal harmonics. Our analysis is based on a simple trick which transforms the corresponding spin-weighted spheroidal angular equation into a Schrödinger-like wave equation which is amenable to a standard WKB analysis. Our *analytical* results for the prolate asymptotic spectrum agree with previous *numerical* computations of the eigenvalues which appear in the literature.

I. INTRODUCTION.

The characteristic dynamics of test fields in black-hole spacetimes has been studied extensively since the pioneering work of Regge and Wheeler [1], see also [2–4] and references therein. An astrophysically realistic model of wave dynamics in black-hole spacetimes should involve a non-spherical background geometry with angular momentum. In terms of the Boyer-Lindquist coordinates, the spacetime of a rotating Kerr black hole is described by the line-element [5, 6]

$$ds^2=-\Big(1-\frac{2Mr}{\rho^2}\Big)dt^2-\frac{4Mar\sin^2\theta}{\rho^2}dtd\phi+\frac{\rho^2}{\Delta}dr^2+\rho^2d\theta^2+\Big(r^2+a^2+\frac{2Ma^2r\sin^2\theta}{\rho^2}\Big)\sin^2\theta d\phi^2, \tag{1}$$

where M and a are the mass and angular momentum per unit mass of the black hole, respectively. (We use gravitational units in which $G=c=1$). Here $\Delta\equiv r^2-2Mr+a^2$ and $\rho\equiv r^2+a^2\cos^2\theta$.

In this paper we consider perturbations of the *non*-spherical Kerr spacetime. The dynamics of a test field Ψ in the rotating Kerr spacetime is governed by the well-known Teukolsky master equation [7]. One may decompose the field as

$${}_s\Psi_{lm}(t,r,\theta,\phi)=e^{im\phi}{}_sS_{lm}(\theta;a\omega){}_s\psi_{lm}(r)e^{-i\omega t}\ , \tag{2}$$

where ω is the (conserved) frequency of the mode, l is the spheroidal harmonic index, and m is the azimuthal harmonic index. The parameter s is called the spin weight of the field, and is given by $s=\pm2$ for gravitational perturbations, $s=\pm1$ for electromagnetic perturbations, $s=\pm\frac{1}{2}$ for massless neutrino perturbations, and $s=0$ for scalar perturbations [7].

With the decomposition (2), ψ and S obey radial and angular equations, both of confluent Heun type [7–11], coupled by a separation constant $A(a\omega)$. The radial Teukolsky equation is given by [7]

$$\Delta^{-s}\frac{d}{dr}\Big(\Delta^{s+1}\frac{d\psi}{dr}\Big)+\Big[\frac{K^2-2is(r-M)K}{\Delta}-a^2\omega^2+2ma\omega-A+4is\omega r\Big]\psi=0\ , \tag{3}$$

where $K\equiv(r^2+a^2)\omega-am$.

The angular functions $S(\theta;a\omega)$ are the spin-weighted spheroidal harmonics which are solutions of the angular equation [7–11]

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial S}{\partial\theta}\Big)+\Big[c^2\cos^2\theta-2cs\cos\theta-\frac{(m+s\cos\theta)^2}{\sin^2\theta}+s+A\Big]S=0\ , \tag{4}$$

where $c\equiv a\omega$. The angular functions are required to be regular at the poles $\theta=0$ and $\theta=\pi$. These boundary conditions pick out a discrete set of eigenvalues $\{{}_sA_{lm}\}$ labeled by the integers m and l. [In the $c\to0$ limit these angular functions become the familiar spin-weighted spherical harmonics with the corresponding angular eigenvalues $A=l(l+1)-s(s+1)+O(a\omega)$.]

The spin-weighted spheroidal harmonics $S(\theta;c)$ and their corresponding eigenvalues $\{{}_sA_{lm}\}$ have attracted much attention over the years from both physicists and mathematicians [11–17]. It is worth emphasizing that in order

to compute the characteristic resonances of black holes [2], one must *first* compute the (closely related) angular eigenvalues $\{{}_sA_{lm}\}$ [see Eq. (3)]. In particular, in the framework of semi-classical general relativity, it has been conjectured that the highly damped resonances may shed light on the quantum properties of black holes [18, 19]. For rotating black holes, these highly damped resonances are characterized by $c_I \to \infty$ (where $c_I \equiv \Im c$).

The asymptotic limit $c_I \to \infty$ of the angular equation (4) was studied in [11, 12] for the $s = 0$ case. It was found that the asymptotic scalar eigenvalues are given by

$$ {}_0A_{lm} = [2(l - |m|) + 1]|c_I| + O(1) \ . \tag{5}$$

As pointed out in [16], the analysis of [11, 12] for the spin-0 case is somewhat incomplete. The analysis of [11, 12] requires the knowledge of the number of zeros of the scalar harmonics in the interval $[0, \pi]$. However, as emphasized in [16], the approximated solution found in [11, 12] for the $c_I \to \infty$ limit is only valid in a region far from the end-points $\theta = 0, \pi$. Thus, the analysis of [11, 12] cannot rule out the possible existence of additional zeros of the angular eigenfunctions near the end-points. The possible omission of such zeros would lead to a wrong asymptotic behavior of the prolate (with $c = ic_I$) eigenvalues, see [16] for details. In this respect the analysis of [11, 12] is not complete. One of the goals of the present paper is to present a more rigorous proof of the formula (5) for the asymptotic scalar eigenvalues.

The asymptotic spectrum of the prolate eigenvalues for the general spin case was first studied in [13]. However, as pointed out in [16, 20], the analysis of [13] for the general spin case is fundamentally flawed (see [16, 20] for details). The *numerical* results presented in Refs. [16, 20] for the general spin case provide evidence for the asymptotic behavior

$$ {}_sA_{lm} = \gamma|c_I| + O(1) \ , \tag{6}$$

where the function γ depends on the spin-parameter s, the azimuthal harmonic index m, and the spheroidal harmonic index l [see Eq. (21) below]. The main aim of the present paper is to provide a (simple) *analytical* proof for the prolate formula (6) in the *general* spin case.

II. A COORDINATE TRANSFORMATION

It proves useful to introduce the coordinate x defined by [17, 21]

$$ x \equiv \ln\Big(\tan\Big(\frac{\theta}{2}\Big)\Big) \ , \tag{7}$$

in terms of which the angular equation (4) becomes a Schrödinger-like wave equation of the form [22]

$$ \frac{d^2S}{dx^2} - US = 0 \ , \tag{8}$$

where

$$ U(\theta) = (m + s\cos\theta)^2 - \sin^2\theta(c^2\cos^2\theta - 2cs\cos\theta + s + A) \ . \tag{9}$$

Note that the interval $\theta \in [0, \pi]$ maps into $x \in [-\infty, \infty]$. The Schrödinger-type angular equation (8) is now in a form that is amenable to a standard WKB analysis.

III. THE SPIN-0 (SCALAR) CASE.

We shall first consider the spin-0 (scalar) case. In this case the effective potential $U(\theta)$ is in the form of a symmetric (invariant under the transformation $\theta \to \pi - \theta$) potential well: in the $c_I \to \infty$ limit it has a local minimum at

$$ {}_0\theta_{\min} = \frac{\pi}{2} \quad \text{with} \quad U({}_0\theta_{\min}) = -A + O(1) \ . \tag{10}$$

Regions where $U(\theta) < 0$ are characterized by an oscillatory behavior of the wave-function S (the 'classically allowed regions'), while regions with $U(\theta) > 0$ (the 'classically forbidden regions') are characterized by an exponential behavior (evanescent waves). The 'classical turning points' are characterized by $U = 0$. There is a pair $\{{}_0\theta^-, {}_0\theta^+\}$ of such turning points (with ${}_0\theta^- < {}_0\theta_{\min} < {}_0\theta^+$), which in the $c_I \to \infty$ limit are located in the near vicinity of ${}_0\theta_{\min}$:

$$ {}_0\theta^{\pm} = \frac{\pi}{2} \pm \frac{\sqrt{A}}{|c_I|} + O(c_I^{-3/2}) \ . \tag{11}$$

A standard textbook second-order WKB approximation for the bound-state 'energies' of a Schrödinger-like wave equation of the form (8) yields the well-known quantization condition [23–27]

$$\int_{x^-}^{x^+} dx\sqrt{-U(x)} = (N+\frac{1}{2})\pi \quad ; \quad N=0,1,2,\dots\ , \tag{12}$$

where $x^{\pm}$ are the turning points [with $U(x^{\pm})=0$] of the potential well, and N is a non-negative integer.

Using the relation $dx/d\theta = 1/\sin\theta$, one can write the WKB condition (12) in the form

$$\int_{{}_0\theta^-}^{{}_0\theta^+} d\theta\frac{\sqrt{-U(\theta)}}{\sin\theta} = (N+\frac{1}{2})\pi \quad ; \quad N=0,1,2,\dots\ . \tag{13}$$

The WKB quantization condition (13) determines the eigenvalues $\{A\}$ of the associated spin-weighted spheroidal harmonics in the large-c_I limit. The relation so obtained between the eigenvalues and the parameters c, m, s and N is rather complex and involves elliptic integrals. However, given the fact that in the $c_I\to\infty$ limit the turning points ${}_0\theta^{\pm}$ lie in the vicinity of $\theta=\frac{\pi}{2}$ [see Eq. (11)], one can approximate the integral in (11) by [28]

$$\int_{{}_0\theta^-}^{{}_0\theta^+} d\theta\sqrt{-c_I^2(\theta-\frac{\pi}{2})^2+A} = (N+\frac{1}{2})\pi \quad ; \quad N=0,1,2,\dots\ . \tag{14}$$

Evaluating the integral in (14) is straightforward, and one finds

$$A(N) = (2N+1)|c_I| + O(1) \quad ; \quad N=0,1,2,\dots \tag{15}$$

for the quantized spectrum. This completes our proof for the prolate asymptotic spectrum in the scalar ($s=0$) case [29].

IV. THE GENERAL SPIN CASE.

We shall now consider the general spin case. In this case the effective potential $U(\theta)$ is complex-valued. Its minimum is located at ${}_s\theta_{\text{min}} = \frac{\pi}{2} + \frac{is}{c_I} + O(c_I^{-2})$, while the two turning points are located at

$${}_s\theta^{\pm} = \frac{\pi}{2} \pm \frac{\sqrt{A}}{|c_I|} + \frac{is}{c_I} + O(c_I^{-3/2})\ . \tag{16}$$

A natural generalization of the WKB analysis to the case of complex-valued potentials is provided in [30]: the WKB quantization rule is given by the standard relation

$$\int_{{}_s\theta^-}^{{}_s\theta^+} d\theta\frac{\sqrt{-U(\theta)}}{\sin\theta} = (N+\frac{1}{2})\pi \quad ; \quad N=0,1,2,\dots\ , \tag{17}$$

which can be approximated near $\theta=\pi/2$ by [28]

$$\int_{{}_s\theta^-}^{{}_s\theta^+} d\theta\sqrt{-c_I^2(\theta-\frac{\pi}{2})^2-2isc_I(\theta-\frac{\pi}{2})+A} = (N+\frac{1}{2})\pi \quad ; \quad N=0,1,2,\dots\ . \tag{18}$$

As emphasized in [30], the integration path between the two complex turning points $\{{}_s\theta^-, {}_s\theta^+\}$ should be chosen such that

$$\Im\{\sqrt{-U(\theta)}\} = 0 \tag{19}$$

along the integration contour [30]. In the $c_I\to\infty$ limit this requirement is easily fulfilled by a straight line (parallel to the real θ-axis) which connects the two turning points (16). Substituting $\theta=\phi-\frac{is}{c_I}$ (where $\phi\in\mathbb{R}$ runs from ${}_0\theta^-$ to ${}_0\theta^+$) into (18) and neglecting terms of order $O(1)$, one finds that along the path (19) the integral (16) can be written as

$$\int_{{}_0\theta^-}^{{}_0\theta^+} d\phi\sqrt{-c_I^2(\phi-\frac{\pi}{2})^2+A} = (N+\frac{1}{2})\pi \quad ; \quad N=0,1,2,\dots\ . \tag{20}$$

This yields

$$A(N) = (2N+1)|c_I| + O(1) \quad ; \quad N = 0, 1, 2, ... \tag{21}$$

for the quantized spectrum. This completes our proof for the prolate asymptotic spectrum in the general spin case [29]. It is worth emphasizing that the *analytical* formula (21) for the prolate asymptotic spectrum agrees with the *numerical* results presented in [16, 20].

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